When Isoparametric met VEM
(VEM for solid mechanics)

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NEMESIS
Kick-off workshop
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Introduction

This is a joint project with:

- U. Perego, M. Cremonesi (Dept of Civil and Environmental Engineering, Politecnico di Milano);
- Abaqus FEA;
- C. Lovadina (Dept of Mathematics, University of Milano);
- F. Dassi (Dept of Mathematics, University of Milano-Bicocca);
- various PhD students

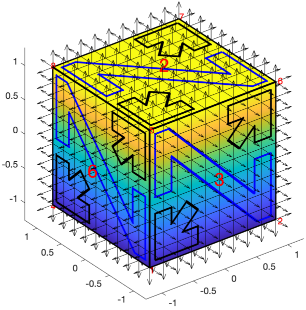
Introduction

AIM of the project:

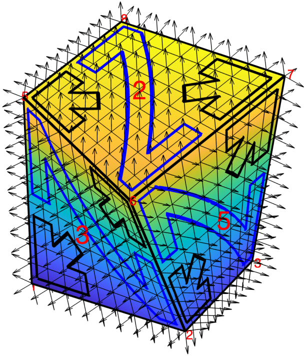
Define a Virtual Element “compatible” with the standard 8-node brick element (Isoparametric $\mathbb{Q}_1$ in 3D) such that:

- it is more robust than FEM with respect to distorsions;
- it works when the distorsion is so large so that the standard 8-node Brick Element does not exists;
- does not need stabilization.

Isoparametric 8-node brick



$$\xrightarrow{\quad \vec{T} \quad}$$



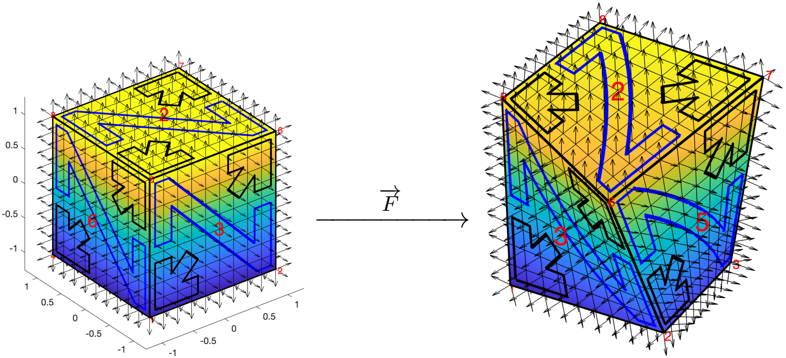
Reference cube $\widehat{B} = [-1, +1]^3$
$(\hat{x}, \hat{y}, \hat{z})$ coordinates

Current element B
(x, y, z) coordinates

$$\vec{T}(\hat{x}, \hat{y}, \hat{z}) = \begin{bmatrix} T_x(\hat{x}, \hat{y}, \hat{z}) \\ T_y(\hat{x}, \hat{y}, \hat{z}) \\ T_z(\hat{x}, \hat{y}, \hat{z}) \end{bmatrix}$$

$T_x(\hat{x}, \hat{y}, \hat{z})$, $T_y(\hat{x}, \hat{y}, \hat{z})$ and $T_z(\hat{x}, \hat{y}, \hat{z})$ are TRILINEAR in $(\hat{x}, \hat{y}, \hat{z})$.

Isoparametric Finite Elements for Bricks



Reference cube $\widehat{B} = [-1, +1]^3$
$(\hat{x}, \hat{y}, \hat{z})$ coordinates

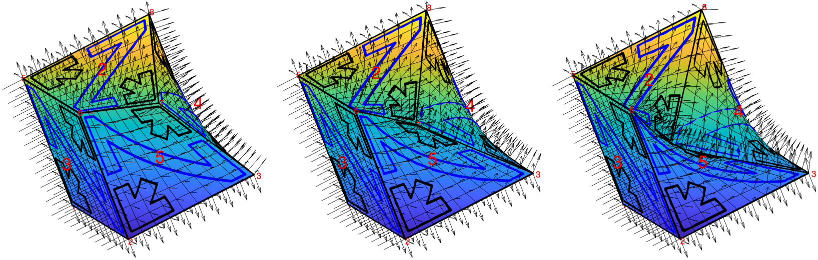
Current element B
(x, y, z) coordinates

There exists a unique trilinear map $\overrightarrow{T}$ that send

$$\text{vertex } i \text{ of } \widehat{B} \quad \longrightarrow \quad \text{vertex } i \text{ of } B$$

If B is a "reasonable perturbation" of a cube, $\overrightarrow{T}$ maps the interior of $\widehat{B}$ in the interior of B and is one-to-one.

Isoparametric Finite Elements for Bricks

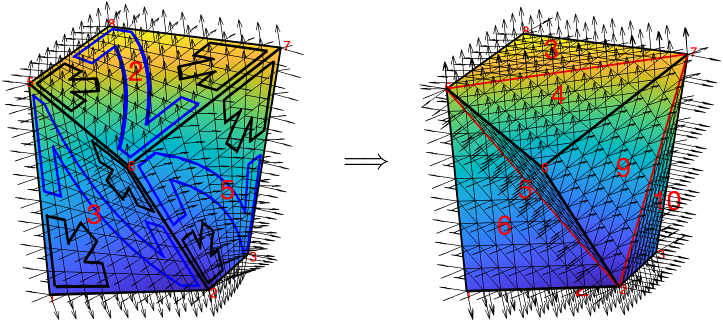


For the last brick, the map $\overrightarrow{T}$ is no more one-to-one and isoparametric Finite Elements do not exist.

Is it possible to define a Virtual Element for a brick like this?

First idea: use deltahedra

The first idea we had was to split (curved) faces into two triangles:



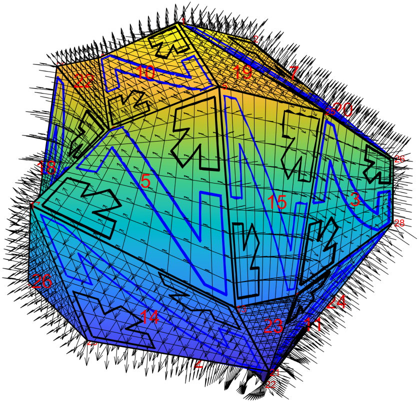
The degrees of freedom remain the same as the 8-node brick's.

Drawbacks:

- the spaces of opposing faces VEM-FEM are not the same...
- there are two possible splittings...

NEW IDEA!

Simply define Virtual Element for a solid like this:

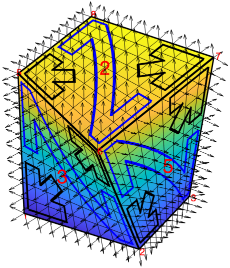


where the faces can be either:

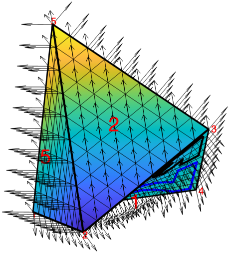
- flat polygons (already done)
- **skew quadrilaterals equal to the (curved) faces of a standard 8-point brick.**

Particular cases:

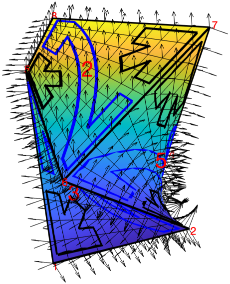
Standard 8-node brick



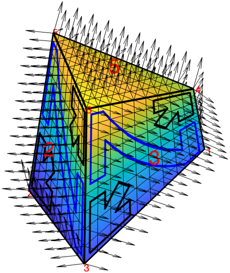
Pyramid



Pretty bad brick



Prysm



The Virtual space on the "skew polyhedron"

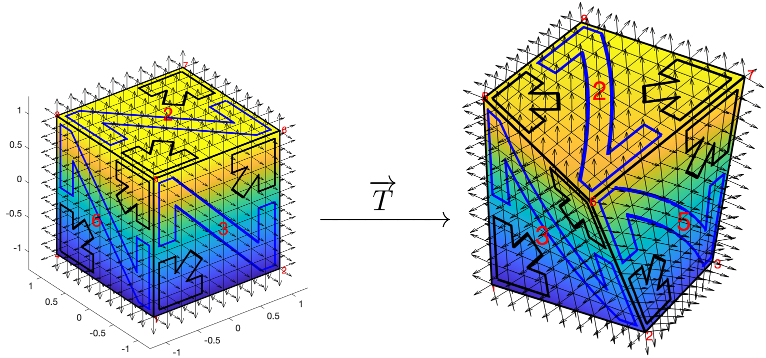
First of all, we definitely need a fancier name instead of "skew polyhedron". Any suggestions from the audience?

The IDEA here is to use isoparametric mappings only for faces and not for the interior of the element.

Plan:

1. define the local space on faces `f`;
2. extend inside with the usual VEM machinery.

The faces of the standard 8-node brick

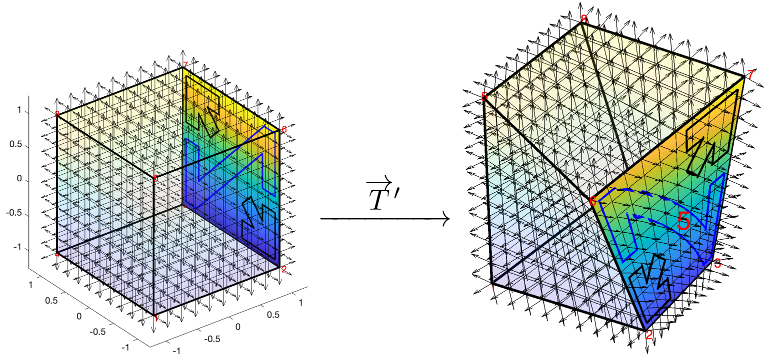


Reference cube $\widehat{B} = [-1, +1]^3$
$(\hat{x}, \hat{y}, \hat{z})$ coordinates

Current element B
(x, y, z) coordinates

The map $\overrightarrow{F} : \widehat{B} \longrightarrow B$ is TRILINEAR in $(\hat{x}, \hat{y}, \hat{z})$

The faces of the standard 8-node brick

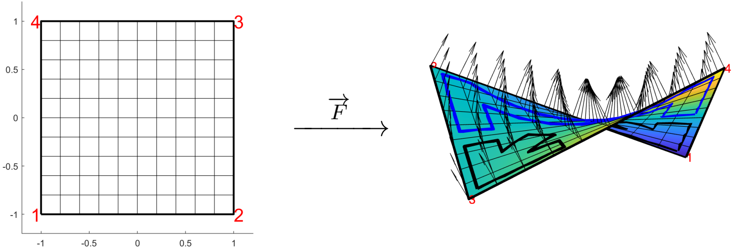


Reference cube $\widehat{B} = [-1, +1]^3$
$(\hat{x}, \hat{y}, \hat{z})$ coordinates

Current element B
(x, y, z) coordinates

The map $\overrightarrow{T}_{|\mathtt{f}_5} = \overrightarrow{T}(-1, \hat{y}, \hat{z}) = \overrightarrow{T}'(\hat{y}, \hat{z})$ is BILINEAR in $(\hat{y}, \hat{z})$

Virtual Elements for "skew polyhedra": the face space



Reference element $\widehat{Q} = [-1,+1] \times [-1,+1]$
(u, v) coordinates

Current face $\mathtt{f}$
(x, y, z) coordinates

There exists a unique bilinear map $\overrightarrow{F} : \widehat{Q} \longrightarrow \mathbb{R}^3$ that sends

$$\text{vertex } i \text{ of } \widehat{Q} \quad \longrightarrow \quad \text{vertex } i \text{ of } \mathtt{f}$$

- We define $\mathtt{f} := \overrightarrow{F}(\widehat{Q})$.
- If the vertices of $\mathtt{f}$ are not co-planar, $\overrightarrow{F}$ maps the interior of $\widehat{Q}$ in the interior of $\mathtt{f}$ and is one-to-one.
- If the vertices of $\mathtt{f}$ are co-planar, $\mathtt{f}$ must be convex.

Virtual Elements for "skew polyhedra": the face space

- In the reference element, the bilinear basis functions are:

$$\widehat{\varphi}_1(u,v) = \frac{1}{4}(1-u)(1-v) \quad \text{and so on}$$

- The map $\overrightarrow{F} : \widehat{Q} \longrightarrow \mathbb{R}^3$ can we written as

$$\overrightarrow{F}(u,v) = \sum_{i=1}^{4} \widehat{\varphi}_i(u,v)\,\overrightarrow{V_i}$$

  where $\overrightarrow{V_i} = (x_i, y_i, z_i)$ are the vertices of the face `f`.

- We write $\overrightarrow{F}$ in components as $\overrightarrow{F} = (F_x, F_y, F_z)$.
- It is well-known that

$$\sum_{i=1}^{4} \widehat{\varphi}_i(u,v) = 1.$$

Virtual Elements for “skew polyhedra”: the face space

Collecting together the relationships written above, we have:

$$\begin{bmatrix} 1 \\ F_x(u,v) \\ F_y(u,v) \\ F_z(u,v) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} \begin{bmatrix} \widehat{\varphi}_1(u,v) \\ \widehat{\varphi}_2(u,v) \\ \widehat{\varphi}_3(u,v) \\ \widehat{\varphi}_4(u,v) \end{bmatrix}.$$

Hence, if the matrix is invertible, the correspondence

$$(F_x, F_y, F_z) \in \mathtt{f} \quad \longleftrightarrow \quad (\widehat{\varphi}_1, \widehat{\varphi}_2, \widehat{\varphi}_3, \widehat{\varphi}_4)$$

is one-to-one.

Virtual Elements for "skew polyhedra": the face space

The invertibility of the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}$$

is equivalent to the fact that the vertices $\overrightarrow{V_i}$ are NOT coplanar.

It easy to see that its determinant is equal to the volume of the pyramid having vertex in $\overrightarrow{V_1}$ and the triangle $\overrightarrow{V_2}\overrightarrow{V_3}\overrightarrow{V_4}$ as basis.

Virtual Elements for "skew polyhedra": the face space

Recalling that the $\widehat{\varphi}_i$'s are barycentric coordinates, i.e. they reproduce linears, we have similarly

$$\begin{bmatrix} 1 \\ u \\ v \\ uv \end{bmatrix} = \begin{bmatrix} +1 & +1 & +1 & +1 \\ -1 & +1 & +1 & -1 \\ -1 & -1 & +1 & +1 \\ -1 & +1 & -1 & +1 \end{bmatrix} \begin{bmatrix} \widehat{\varphi}_1(u,v) \\ \widehat{\varphi}_2(u,v) \\ \widehat{\varphi}_3(u,v) \\ \widehat{\varphi}_4(u,v) \end{bmatrix}.$$

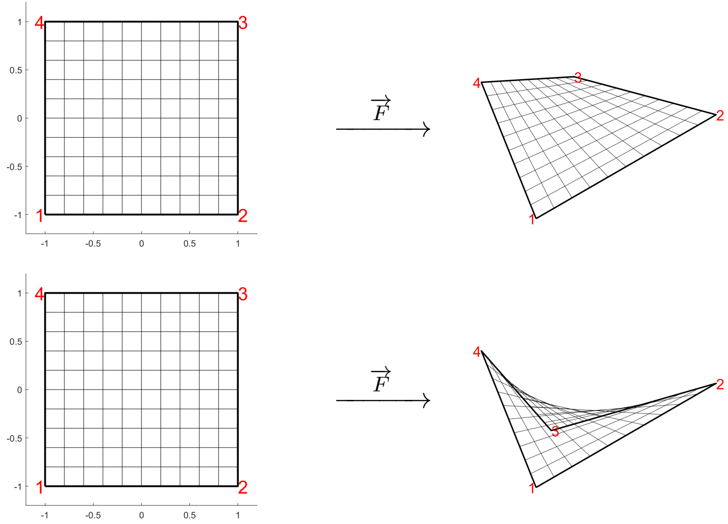
where the determinant of the matrix is $4\times$(area of $\widehat{Q}$) (i.e. $= 8$).

Hence, if we start from $(F_x(u,v), F_y(u,v), F_z(u,v)) \in \mathtt{f}$, we get a unique vector $(1, u, v, uv)$ which implies a unique $(u,v) \in \widehat{Q}$.

We conclude that if the vertices $\overrightarrow{V_i}$ are NOT coplanar, then the map $\overrightarrow{F}$ is one-to-one from $\widehat{Q}$ to $\mathtt{f}$.

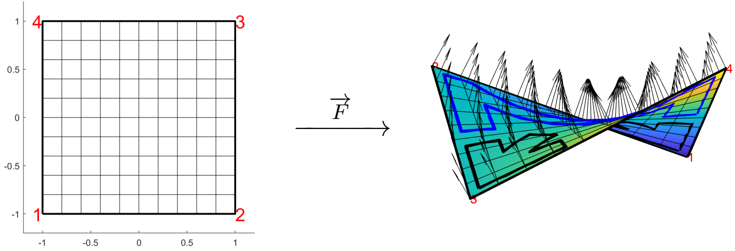
The case of coplanarity corresponds to the isoparametric bilinear map in two dimensions:

Virtual Elements for “skew polyhedra”: the face space



The map $\vec{F}$ is no more one-to-one.

Virtual Elements for "skew polyhedra": the face space



Reference element $\widehat{Q} = [-1, +1] \times [-1, +1]$
(u, v) coordinates

Current face $\mathtt{f}$
(x, y, z) coordinates

A function $\psi : \mathtt{f} \longrightarrow \mathbb{R}$ belongs to the space $V_1(\mathtt{f})$ of the face $\mathtt{f}$ if it is "bilinear in the parameters", i.e. if the composed function

$$(\psi \circ \overrightarrow{F})(u, v) = \psi\big(F_x(u, v), F_y(u, v), F_z(u, v)\big)$$

is bilinear in (u, v).

The face space $V_1(\mathtt{f})$ contains linear polynomials

Lemma

If $p_1(x, y, z)$ is a linear polynomial in three variables, its restriction to $\mathtt{f}$ belongs to the space $V_1(\mathtt{f})$.

Proof:

By definition of $V_1(\mathtt{f})$, we have to show that

$$(p_1 \circ \overrightarrow{F})(u, v) = p_1\big(F_x(u, v), F_y(u, v), F_z(u, v)\big)$$

is bilinear in (u, v).

Assume that $p_1(x, y, z) = a + bx + cy + dz$; then

$$(p_1 \circ \overrightarrow{F})(u, v) = a + b\,F_x(u, v) + c\,F_y(u, v) + d\,F_z(u, v)$$

IS bilinear since F_x, F_y, F_z are bilinear. □

The face space $V_1(\mathtt{f})$ coincides with linear polynomials?

- The space of linear polynomials in three variables $\mathbb{P}(x, y, z)$ has dimension 4.
- The dimension of $V_1(\mathtt{f})$ is also 4, since a basis is given by the basis functions of the reference element mapped through $\overrightarrow{F}^{-1}$

$$V_1(\mathtt{f}) = \text{span}\,\{(\overrightarrow{F})^{-1} \circ \widehat{\varphi}_i\}$$

- It is clear that when the face $\mathtt{f}$ is flat (coplanar vertices), then dimension of the restrictions to $\mathtt{f}$ of linear polynomials in (x, y, z) drops down to 3 and correspond to linear polynomials in two variables.

- What happens when the face $\mathtt{f}$ is NOT flat?

The face space $V_1(\mathtt{f})$ coincides with linear polynomials?

Lemma

If the vertices of $\mathtt{f}$ are NOT coplanar, then the restrictions of linear polynomials in (x, y, z) to $\mathtt{f}$ coincide with $V_1(\mathtt{f})$.

Proof:

We need just to understand when it happens that:

- restricting $\{1, x, y, z\}$ to $\mathtt{f}$, they remain linearly independent.

Equivalently, we can check when

$$\{1 \circ \overrightarrow{F},\ x \circ \overrightarrow{F},\ y \circ \overrightarrow{F},\ z \circ \overrightarrow{F}\}$$

defined on $\widehat{Q}$ are linearly independent.

Since $1 \circ \overrightarrow{F} = 1$, $x \circ \overrightarrow{F} = F_x$ and so on, we have to check when

$$\{1,\ F_x(u, v),\ F_y(u, v),\ F_z(u, v)\}$$

are linearly independent.

The face space $V_1(\mathtt{f})$ coincides with linear polynomials?

From the definition of $\overrightarrow{F}$ we know that

$$F_x(u,v) = \sum_{i=1}^{4} \widehat{\varphi}_i(u,v) x_i \quad \text{and so on}$$

that (together with $\sum_{i=1}^{4} \widehat{\varphi}_i = 1$) we can write in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} \begin{bmatrix} \widehat{\varphi}_1 \\ \widehat{\varphi}_2 \\ \widehat{\varphi}_3 \\ \widehat{\varphi}_4 \end{bmatrix} = \begin{bmatrix} 1 \\ F_x \\ F_y \\ F_z \end{bmatrix}.$$

Since the determinant of the matrix is NOT zero if and only if the vertices $\overrightarrow{V_i}$ are NOT coplanar, the Lemma is proved. □

Linear Virtual Element for the "skew polyhedron" $\mathcal{S}$

We consider the easiest case, i.e. the scalar equation

$$-\operatorname{div}(\boldsymbol{D}\nabla u) = 0.$$

- For each face $\mathtt{f}$ we have defined a space $V_1(\mathtt{f})$ that contains (restriction of) linear polynomials.
- We can define the space $V_1(\mathcal{S})$ in the usual way:

$$V_1(\mathcal{S}) = \{v_h : \mathcal{S} \longrightarrow \mathbb{R} \quad \text{such that :}$$
$$\bullet\, v_{h|\mathtt{f}} \in V_1(\mathtt{f}) \text{ for all faces } \mathtt{f};$$
$$\bullet\, v_h \text{ on the boundary of } \mathcal{S} \text{ is continuous;}$$
$$\bullet\, v_h \text{ is harmonic in } \mathcal{S}, \text{ i.e. } \Delta v_h = 0\}$$

- The space $V_1(\mathcal{S})$ contains the linear polynomials in (x, y, z);
- the degrees of freedom of v_h are the pointwise values at the vertices.

Linear Virtual Element for the "skew polyhedron" $\mathcal{S}$

The Π^∇ operator is a projection from the space $V_1(\mathcal{S})$ to the space of linear polynomials $\mathbb{P}_1(x,y,z)$ defined in the following way:

$$\begin{cases} \int_{\mathcal{S}} \nabla[\Pi_1^\nabla v_h] \cdot \nabla p_1 \,\mathrm{d}x = \int_{\mathcal{S}} \nabla v_h \cdot \nabla p_1 \,\mathrm{d}x \\ P_0[\Pi_1^\nabla v_h] = P_0 v_h \end{cases}$$

where for instance $P_0\psi = \frac{1}{N_V}\sum_{i=1}^{N_V} \psi(V_i)$.

Since the gradient of p_1 is a constant vector, in order to compute $\Pi_1^\nabla \varphi_i$ we only need to compute the mean value of $\nabla\varphi_i$:

$$\frac{1}{|\mathcal{S}|}\int_{\mathcal{S}} \nabla\varphi_i \,\mathrm{d}x = \int_{\partial\mathcal{S}} \varphi_i \boldsymbol{n} \,\mathrm{d}s = \sum_{\mathtt{f}} \int_{\mathtt{f}} \varphi_i \boldsymbol{n}_{\mathtt{f}} \,\mathrm{d}s =$$

$$= (\text{for } \mathtt{f} \text{ skew}) = \sum_{\mathtt{f}} \int_{-1}^{+1}\int_{-1}^{+1} \underbrace{\widehat{\varphi}_\ell^{\mathtt{f}}(u,v) \left[\frac{\partial \overrightarrow{F}^{\mathtt{f}}}{\partial u} \times \frac{\partial \overrightarrow{F}^{\mathtt{f}}}{\partial v}\right]}_{4^{\text{th}} \text{ order polynomial (explicit)}} \mathrm{d}u\mathrm{d}v$$

Linear Virtual Element for the "skew polyhedron" $\mathcal{S}$

Once we have the mean value of $\nabla\varphi_i$, we can compute $\Pi_1^\nabla\varphi_i$:

$$[\Pi_1^\nabla\varphi_i](\boldsymbol{x}) = \left(\frac{1}{|\mathcal{S}|}\int_{\mathcal{S}}\nabla\varphi_i\,\mathrm{d}x\right)\cdot(\boldsymbol{x}-\bar{\boldsymbol{x}}) + \frac{1}{N_V}$$

and therefore the VEM consistency and stabilization matrices.

A quadrature formula exact for linear polynomials with the vertices as quadrature nodes can be readily obtained by interpolating in the space $V_1(\mathcal{S})$ and then approximating the integrand with its Π_1^∇ projection:

$$\psi \approx \psi_I = \sum_{i=1}^{N_V}\psi(V_i)\varphi_i$$

$$\int_{\mathcal{S}}\psi\,\mathrm{d}x \approx \int_{\mathcal{S}}\psi_I\,\mathrm{d}x \approx \int_{\mathcal{S}}\Pi_1^\nabla\psi_I\,\mathrm{d}x = \sum_{i=1}^{N_V}\overset{\text{nodes}}{\psi(V_i)}\underbrace{\int_{\mathcal{S}}\Pi_1^\nabla\varphi_i\,\mathrm{d}x}_{\text{weigths}}$$

If ψ is a linear polynomials, all $\boxed{\approx}$ become $\boxed{=}$.

Conclusions

- We have defined a Virtual Element on a "skew polyhedron", i.e. a polyhedron with faces that can be either flat polygons or skew quadrilaters;
- the face space is the same used in classical 8-node brick, hence these new Virtual Elements is perfectly compatible with the classical isoparametric 8-node brick;
- the Virtual Element con be employed for "deformed" 8-node bricks for which the Jacobian of the transformation becomes singular;
- they show a remarkable robustness with respect to large deformations;
- the same idea be readily extended to the high-order case, employing the $\mathbb{Q}_k$ polynomials on the faces and standard VEM of order k inside;
- in principle they admit a stabilization-free approach (to be checked).